Int. J. Heat Mass Transfer. Vol. 10, pp. 1171–1178. Pergamon Press Ltd. 1967. Printed in Great Britain

FREE-CONVECTION STAGNATION FLOW OF AN ABSORBING–EMITTING GAS

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(*Received* 9 *August* 1966 *and in revised form* 14 *January* 1967)

Abstract—An analysis is presented for laminar free convection of an absorbing–emitting gas in the region of the stagnation point of a horizontal cylinder. The analysis is formulated for a gray gas and a black isothermal cylinder. The parameters which govern the relative role of radiation to conduction and convection are varied over a range of representative values. The analysis has been limited to a Prandtl number of one.

NOMENCLATURE

a, absorption coefficient;
c_p, specific heat at constant pressure;
$E_n(t)$, exponential integral, $\int_0^1 \mu^{n-2} \exp(-t/\mu)\, d\mu$;
f, dimensionless stream function, $\dfrac{\psi R}{\nu x Gr^{\frac{1}{4}}}$;
g, acceleration of gravity;
Gr, Grashof number, $\dfrac{g\beta(|T_w - T_\infty|)R^3}{\nu^2}$;
k, thermal conductivity;
N, $ka/4\sigma T_\infty^3$;
Nu, Nusselt number, $\dfrac{qR}{k(T_w - T_\infty)}$;
Pr, Prandtl number, $\mu c_p/k$;
q_c, conductive heat-transfer rate;
q_r, radiative heat-transfer rate;
q, total heat-transfer rate;
R, radius of cylinder;
t, dummy variable of integration;
T, absolute temperature;
u, velocity component in x-direction;
v, velocity component in y-direction;
x, coordinate along cylinder wall;
y, coordinate normal to cylinder wall.

Greek symbols

β, coefficient of thermal expansion;
γ, T_w/T_∞;
ζ, $aR/Gr^{\frac{1}{4}}$;
η, $(y/R)Gr^{\frac{1}{4}}$;
θ, dimensionless temperature difference, $\dfrac{T - T_\infty}{T_w - T_\infty}$;
μ, dynamic viscosity;
ν, kinematic viscosity;
ξ, $\dfrac{4aR^2\sigma T_\infty^3}{\rho c_p Gr^{\frac{1}{2}}\nu}$;
ρ, density;
σ, Stefan–Boltzmann constant;
τ, optical distance, ay.

Subscripts

w, cylinder surface;
∞, ambient.

INTRODUCTION

THE INTERACTION of thermal radiation with convection heat transfer has recently attracted considerable attention. Except for an investigation of laminar free convection from a vertical flat plate [1], the previous investigations have dealt with forced convection. It can be demonstrated from the results of Cess [1] that radiation interaction has a significant effect on free convection heat transfer.

This investigation considers laminar free convection of an absorbing–emitting gas in the neighborhood of the stagnation point of a horizontal cylinder. In the particular situation studied, the only source of radiation incident on the surface of the cylinder originates from the fluid which is infinite in extent. Although the restrictive assumptions of a non-scattering gray gas, constant properties and a black wall were employed, this investigation serves as an excellent example for examining radiation-interaction effects in boundary layers.

ANALYSIS

Governing equations

The physical model is illustrated in Fig. 1. It is assumed that the flow is steady and laminar

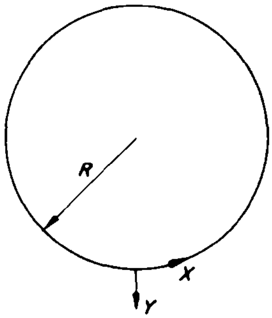

FIG. 1. Physical model.

and, except for the buoyancy term, that the physical properties of the fluid are constant. The wall of the cylinder is black and at a uniform temperature. The absorbing–emitting fluid is a non-scattering diffuse gray gas with an index of refraction of one. In addition, the analysis has been limited to the stagnation region of the cylinder and to cylinders of small curvature.

In light of the assumptions, the conservation equations take the following form*:

* These equations are written in terms of the lower stagnation region of a heated cylinder. For flow in the upper stagnation region of a cooled cylinder, the only modification necessary is a minus sign in the buoyancy term of the momentum equation.

Mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{gx}{R}\frac{(\rho_\infty - \rho)}{\rho} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Energy

$$v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} \tag{3}$$

where the radiative heat transfer in the y-direction is given by [2]

$$q_r = 2\sigma\Big[T_w^4 E_3(\tau) + \int_0^\tau T^4 E_2(\tau - t)\,dt - \int_\tau^\infty T^4 E_2(t - \tau)\,dt\Big]. \tag{4}$$

For simplicity, the majority of the results will be limited to small temperature differences ($T_w/T_\infty \to 1{\cdot}0$). Thus, the linear equations are developed in the main body of the report and the corresponding nonlinear equations are given in the Appendix. Introducing the relation

$$T^4 = 4T_\infty^3(T - T_\infty) + T_\infty^4$$

into equation (4), the linearized radiative heat transfer is given by

$$\frac{q_r}{\sigma(T_w^4 - T_\infty^4)} = 2E_3(\tau) + 2\int_0^\tau \theta(t)\,E_2(\tau - t)\,dt - 2\int_\tau^\infty \theta(t)\,E_2(t - \tau)\,dt. \tag{5}$$

Employing the derivative of equation (5) and the usual transformation, one obtains for equations (2) and (3)*

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 + \theta = 0 \tag{6}$$

* Equations (6) and (7) apply to the upper stagnation region of a cooled cylinder and the lower stagnation region of a heated cylinder.

$$\frac{d^2\theta}{d\eta^2} + Pr\,f\frac{d\theta}{d\eta} + 2Pr\,\xi\Big[E_2(\tau) + \int_0^\infty \theta(t)\,E_1(|\tau - t|)\,dt - 2\theta\Big] = 0. \tag{7}$$

Except for the additional radiation term, the equations are the normal laminar free-convection equations in the stagnation region of a horizontal cylinder [3]. The boundary conditions are

$$\eta, \tau = 0: \quad \theta = 1{\cdot}0, \; f = \frac{df}{d\eta} = 0$$

$$\eta, \tau \to \infty: \quad \theta \to 0, \; \frac{df}{d\eta} \to 0.$$

The parameter ξ which appears in equation (7) denotes the relative role of radiation to convection. The relationship between τ and η is

$$\tau = \frac{aR}{Gr^{\frac{1}{4}}}\eta = \zeta\eta$$

where ζ is a measure of the opacity of the boundary layer [2]. An additional parameter which is employed in radiation–conduction interaction problems is N which denotes the relative role of conduction to radiation. This, however, is not a separate parameter since

$$\zeta^2 = Pr\,\xi\,N.$$

Although the optically thick boundary layer is not normally encountered except under unusual circumstances, it is still of interest to consider the optically thick approximation for sake of completeness. Expanding θ in a series about $t = \tau$

$$\theta(t) = \theta(\tau) + \frac{d\theta(\tau)}{d\tau}(t - \tau) + \ldots$$

and substituting this expression in equation (5), one obtains the following relation

$$\frac{q_r}{\sigma(T_w^4 - T_\infty^4)} = 2E_3(\tau)\,[1 - \theta(\tau)] - 2\frac{d\theta(\tau)}{d\tau}\,[\tfrac{2}{3} - e^{-\tau} + 2E_4(\tau)] + \ldots. \tag{8}$$

In the limit as ζ becomes very large and noting that for large t, $E_n(t) \sim e^{-t}/t$, one obtains for the linearized radiant heat transfer in the region far from the wall*

$$\frac{q_r}{\sigma(T_w^4 - T_\infty^4)} = -\frac{4}{3}\frac{d\theta}{d\tau}. \tag{9}$$

Using the derivative of equation (9) in the energy equation, one obtains

$$\frac{1}{Pr}\left(1 + \frac{4}{3N}\right)\frac{d^2\theta}{d\eta^2} + f\frac{d\theta}{d\eta} = 0. \tag{10}$$

Equations (6) and (10) with $(1 + 4/3N)\,1/Pr$ replaced by $1/Pr$ combined with the previously given boundary conditions describe the non-radiating free-convection problem. Results for this problem exist in the literature for a limited Prandtl number range [4, 5].

Heat transfer

Considering the conduction contribution first, it can be expressed in dimensionless form as

$$\frac{Nu}{Gr^{\frac{1}{4}}} = \frac{q_{cw}R}{k(T_w - T_\infty)\,Gr^{\frac{1}{4}}} = -\left.\frac{d\theta}{d\eta}\right)_{\eta=0}. \tag{11}$$

The dimensionless radiative flux at the wall is obtained by evaluating equation (5) at $\tau = 0$

$$\frac{q_{rw}}{\sigma(T_w^4 - T_\infty^4)} = 1 - 2\int_0^\infty \theta(t)\,E_2(t)\,dt. \tag{12}$$

To properly observe the behavior of q_w in the limits of large and small ξ, it is convenient to write the total heat flux in two different dimensionless forms [2]. As the magnitude of ξ decreases, the role of convection increases and the problem reduces to the non-radiating free-convection problem. This suggests a dimensionless form similar to equation (11)

* The consequences of this approximation will be discussed during the presentation of the numerical results.

$$\frac{q_w R}{k(T_w - T_\infty)\,Gr^{\frac{1}{4}}} = -\left.\frac{d\theta}{d\eta}\right)_{\eta=0} + \frac{Pr\xi}{\zeta}\left[1 - 2\int_0^\infty \theta(t)\,E_2(t)\,dt\right]. \quad (13)$$

A relation which is more useful for describing the behavior of q_w for large values of ξ where radiation dominates the heat-transfer process is given by

$$\frac{q_w}{\sigma(T_w^4 - T_\infty^4)} = -\left.\frac{\zeta}{Pr\xi}\frac{d\theta}{d\eta}\right)_{\eta=0} + 1 - 2\int_0^\infty \theta(t)\,E_2(t)\,dt. \quad (14)$$

For the thick approximation, equations (12–14) reduce to

$$\frac{q_{rw}}{\sigma(T_w^4 - T_\infty^4)} = -\left.\frac{4}{3\zeta}\frac{d\theta}{d\eta}\right)_{\eta=0} \quad (15)$$

$$\frac{q_w R}{k(T_w - T_\infty)\,Gr^{\frac{1}{4}}} = -\left.\left(1 + \frac{4}{3N}\right)\frac{d\theta}{d\eta}\right)_{\eta=0} \quad (16)$$

$$\frac{q_w}{\sigma(T_w^4 - T_\infty^4)} = -\left.\left(\frac{\zeta}{Pr\xi} + \frac{4}{3\zeta}\right)\frac{d\theta}{d\eta}\right)_{\eta=0}. \quad (17)$$

Method of solution

The simultaneous solution of equations (6) and (7) was accomplished by an iterative forward-integration procedure on a UNIVAC 1107. In order to remove the difficulty that $E_1(t)$ has a logarithmic singularity at $t = 0$, the radiation term in equation (7) was first integrated by parts. The starting values for forward integration were obtained by a method outlined by Nachtsheim and Swigert [6]. The integration was repeated until successive iterations yielded $df/d\eta$ distributions within 0·0001.

RESULTS AND DISCUSSION

Results are presented for ζ values ranging from 0·05 to 10·0 and ξ values ranging from 0·001 to 10·0 for a Prandtl number of one. Using values of the Planck mean absorption coefficient given in [2], one finds that the above ranges for ζ and ξ include values that could be reproduced without too much difficulty in the laboratory ($\zeta, \xi \sim 0{\cdot}1$). It should be pointed out that when speaking of a physical situation one has to be careful of curvature effects due to the increased penetration of the thermal layer.

Figure 2 represents the effect of radiation interaction on the conductive component of the total heat transfer at the wall. For $\zeta \geqslant 1{\cdot}0$, the results presented in the figure show an increase in conduction. Due to the slope of the radiative flux being positive near the wall, the radiative term in equation (3) behaves as a heat sink thus increasing conduction. For $\zeta < 1{\cdot}0$, the presented results indicate a minimum in the conductive flux. For small ξ, the effect of the thickening of the boundary layer is greater

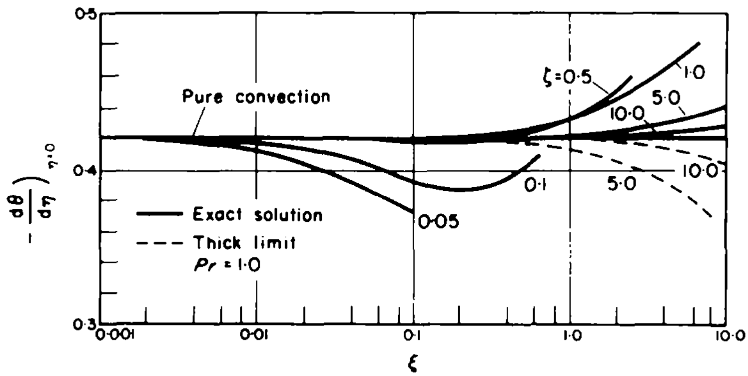

FIG. 2. The effect of radiation interaction on the conductive wall heat flux.

than the sink effect thus causing an initial decrease in $-\,d\theta/d\eta)_{\eta=0}$. It is interesting to note that the thick approximation predicts the incorrect trend in the wall temperature gradient.

The dimensionless radiative component of the total heat transfer is given in Fig. 3 as a function of ζ and ξ. The radiation transfer from the wall decreases with increasing opacity of the gas layer and as the role of radiation increases. The latter is due to the increased penetration of the thermal layer which acts as a radiation shield. Again, it is interesting to note the poor correspondence between the exact solution and thick approximation; the thick approximation overestimates the radiant flux by approximately a factor of two.

The reason for the poor performance of the thick approximation can be obtained by examination of equation (8). The quantity $q_r/\sigma(T_w^4 - T_\infty^4)$ is well represented by $-\frac{4}{3}\,d\theta/d\tau$ in the outer portion of the boundary layer and drops to *approximately* $-\frac{2}{3}\,d\theta/d\tau$ at the wall. Thus, near the wall, the thick approximation predicts the incorrect sign for the radiative term in equation (3) explaining the incorrect behavior of the conductive flux.

Figures 4 and 5 show the effect of radiation interaction on the total heat transfer at the wall.

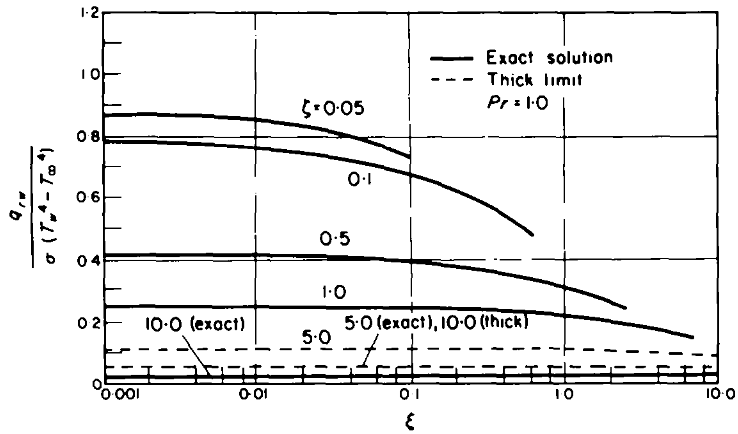

FIG. 3. The effect of radiation interaction on the radiative wall heat flux.

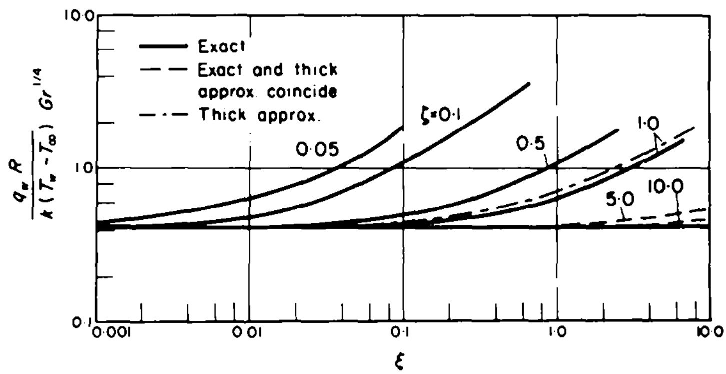

FIG. 4. The effect of radiation interaction on the dimensionless total wall heat flux.

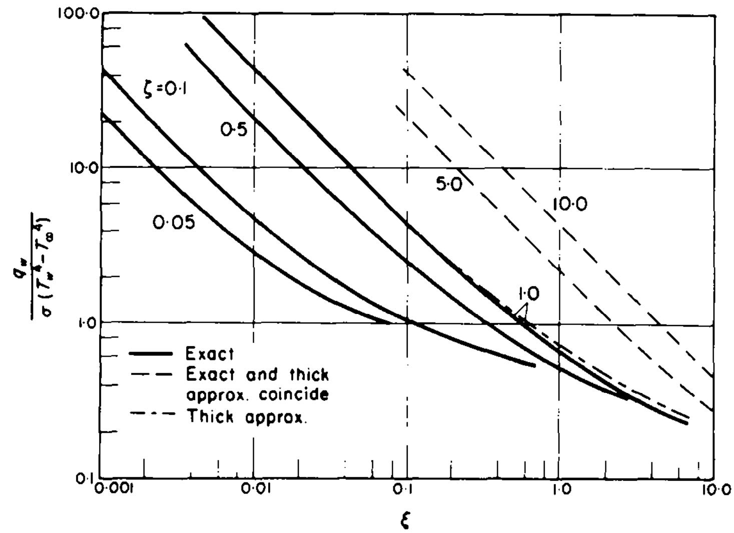

FIG. 5. The effect of radiation interaction on the dimensionless total wall heat flux.

Figure 4 clearly presents the behavior of the total heat transfer for small ξ. In agreement with equations (7) and (13), the total heat-transfer results approach pure free convection as ξ decreases. Figure 5 is a better representation of the total heat transfer for large values of ξ where radiation is the dominant component. Presented in this form, the total heat transfer decreases and the individual ζ curves converge as ξ increases. Although the thick approximation fails to predict the separate components of the total heat transfer, one observes from Figs. 4 and 5 that this approximation gives an excellent prediction of the total heat transfer for large values of ζ. This observation suggests that the thick approximation, as in the case of conduction–radiation interaction problems [7], is a correct asymptotic limit for the total heat transfer in convection–radiation interaction problems. This fact is substantiated analytically in reference [8] using a number of convection–radiation interaction situations as examples.

Figures 6 and 7 illustrate the effects of nonlinear radiation on the individual heat-transfer components for $\zeta = 0{\cdot}5$. The conductive component of the total heat transfer, Fig. 6,

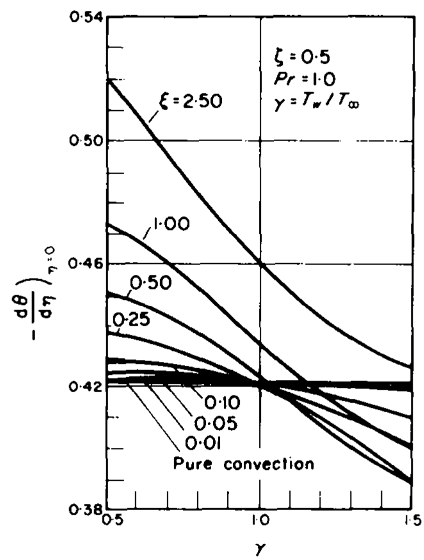

FIG. 6. The dimensionless conductive wall heat flux as a function of T_w/T_∞ for $\zeta = 0{\cdot}5$.

behaves as a function of γ in a manner similar to the optically thin results found for forced convection over a flat plate [9]. As would be expected, the effect of γ increases as the participation of radiation increases. The influence of γ is as large or larger in most cases than the

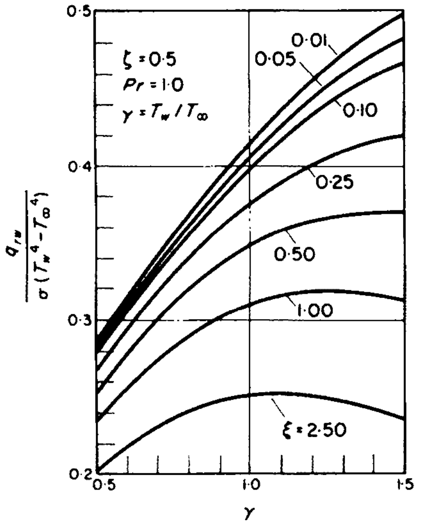

FIG. 7. The dimensionless radiative wall heat flux as a function of T_w/T_∞ for $\zeta = 0{\cdot}5$.

influence of interaction relative to pure free convection. Figure 7 presents the radiative component of the total heat transfer as a function of γ. As before, the trends, except for the maximum which exists at the larger values of ξ, match those found for forced convection in [9]. The influence of γ on the total heat transfer can be obtained by combining the results of Figs. 6 and 7 with equations (21) or (22). Although nonlinear results were obtained for only one value of ζ, the importance of the nonlinear nature of radiation in interaction situations is clearly shown.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the able assistance of W. P. Schimmel in portions of this work and the many helpful suggestions of Dr. K. T. Yang.

REFERENCES

1. R. D. CESS, The interaction of thermal radiation with free convection heat transfer, *Int. J. Heat Mass Transfer*, 9, 1269 (1966).
2. E. M. SPARROW and R. D. CESS, *Radiation Heat Transfer*. Wadsworth, Belmont, Calif. (1966).
3. R. HERMANN, Heat transfer by free convection from horizontal cylinders in diatomic gases, NACA Tech. Mem. 1366 (1954).
4. J. C. Y. KOH and J. F. PRICE, Laminar free convection from a nonisothermal cylinder, *J. Heat Transfer* **87C**, 237 (1965).
5. T. CHIANG and J. KAY, On laminar free convection from a horizontal cylinder, in *Proceedings of the 4th U.S. National Congress of Applied Mechanics*. Berkeley, Calif. (1962).
6. P. R. NACHTSHEIM and P. SWIGERT, Satisfaction of asymptotic boundary conditions in the numerical solution of boundary-layer equations, in *Proceedings of the 9th Midwestern Mechanics Conference*, Madison, Wis. (1965).
7. L. S. WANG, Differential methods for combined radiation and conduction, Ph.D. thesis, University of California, Berkeley (1965).
8. J. L. NOVOTNY and K. T. YANG, The interaction of thermal radiation in optically thick boundary layers, University of Notre Dame TR No. 66-13, 1966. To appear in *J. Heat Transfer*.
9. R. D. CESS, Radiation effects upon boundary-layer flow of an abosrbing gas, *J. Heat Transfer* **86C**, 469 (1964).

APPENDIX

The nonlinear radiation equations corresponding to equations (5), (7), (12), (13) and (14) are

$$\frac{q_r}{\sigma(T_w^4 - T_\infty^4)} = \frac{2}{(\gamma^4 - 1)}\left\{\gamma^4 E_3(\tau) + \int_0^\tau [\theta(\gamma - 1) + 1]^4 E_2(\tau - t)\,\mathrm{d}t - \int_\tau^\infty [\theta(\gamma - 1) + 1]^4 E_2(t - \tau)\,\mathrm{d}t\right\} \tag{18}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} + Pr\frac{f\,\mathrm{d}\theta}{\mathrm{d}\eta} + \frac{Pr\xi}{2(\gamma - 1)}\left\{\gamma^4 E_2(\tau) + \int_0^\infty [\theta(\gamma - 1) + 1]^4 E_1(|\tau - t|)\,\mathrm{d}t - 2[\theta(\gamma - 1) + 1]^4\right\} \tag{19}$$

$$\frac{q_{rw}}{\sigma(T_w^4 - T_\infty^4)} = \frac{1}{(\gamma^4 - 1)} \times \left\{\gamma^4 - 2\int_0^\infty [\theta(\gamma - 1) + 1]^4 E_2(t)\,\mathrm{d}t\right\} \tag{20}$$

$$\frac{q_w R}{k(T_w - T_\infty)\,Gr^{\frac{1}{4}}} = -\left.\frac{d\theta}{d\eta}\right)_{\eta=0} + \frac{Pr\xi}{4\zeta(\gamma - 1)}\left\{\gamma^4 - 2\int_0^\infty [\theta(\gamma - 1) + 1]^4 \times E_2(t)\,dt\right\} \tag{21}$$

$$\frac{q_w}{\sigma(T_w^4 - T_\infty^4)} = -\frac{4\zeta(\gamma - 1)}{\xi Pr(\gamma^4 - 1)}\left.\frac{d\theta}{d\eta}\right)_{\eta=0} + \frac{1}{(\gamma^4 - 1)}\left\{\gamma^4 - 2\int_0^\infty [\theta(\gamma - 1) + 1]^4 \times E_2(t)\,dt\right\}. \tag{22}$$

The above equations apply to the upper stagnation region of a cooled cylinder ($\gamma < 1{\cdot}0$) and the lower stagnation region of a heated cylinder ($\gamma > 1{\cdot}0$). As in the linear case, the radiation term in equation (19) was first integrated by parts to remove $E_1(t)$. It should be pointed out that when numerically integrating equation (18) to a large but finite value of τ, care must be exercised to be sure that all contributions from the second integral are taken into account.

Résumé— On présente une théorie de la convection naturelle laminaire d'un gaz absorbant et émetteur au voisinage du point d'arrêt d'un cylindre chauffé horizontal. La théorie est formulée pour un gaz gris et un cylindre noir isotherme. On fait varier les paramètres gouvernant le rôle relatif du rayonnement par rapport à la conduction et à la convection dans une gamme de valeurs représentatives. On a limité la théorie à un nombre de Prandtl égal à l'unité.

Zusammenfassung—Für die laminare freie Konvektion eines absorbierenden–emittierenden Gases im Bereich des Staupunktes eines beheizten waagerechten Zylinders wird eine Analyse gegeben. Diese Analyse beruht auf der Annahme eines grauen Gases und eines schwarzen isothermen Zylinders. Die für das Verhältnis von Strahlung zu Leitung und Konvektion entscheidenden Parameter wurden im Bereich repräsentativer Werte variiert. Die Analyse ist auf die Prandtl-Zahl eins beschränkt.

Аннотация—Дается анализ для ламинарной свободной конвекции поглощающего–излучающего газа вблизи передней критической точки нагретого горизонтального цилиндра. Задача сформулирована для серого газа и черной изотермической стенки цилиндра и числа $Pc = 1$. Параметры, характеризующие вклад теплового излучения по сравнению с теплопроводностью и конвекцией, варьируются в широком диапазоне соответствующих значений.